International Journal of Theoretical Physics, Vol. 7, No. 6 (1973), pp. 451–466

Some Problems of Quantum Mechanics Possessing a Non-Negative Phase-Space Distribution Function

V. V. KURYSHKIN

Equipe de Recherche sur les Fondements de la Physique Quantique, Académie des Sciences, 3 rue Mazarine, Paris 6e, France†

Received: 3 *July* 1972

Abstract

In order to clarify physical consequences due to the presence of a set of auxiliary functions $\phi_k(q,t)$ in quantum mechanics with a non-negative phase-space distribution function, the simplest quantum-mechanical problems are solved. It is shown that $\phi_k(q,t)$ influence upon the results of a problem. Therefore it is supposed that $\phi_k(q,t)$ reflect some physical reality (subquantum situation), interacting with a mechanical system. In particular the 'subquantum situation' determines the minimum coordinate and momentum uncertainties ($(\delta q)^2$ and $(\delta p)^2$) as well as the coordinate distribution of a 'fixed' system and the momentum distribution of a 'free' system. These results provide the opportunity to formulate the notion of a stationary homogeneous isotropic 'subquantum situation'. Supposing that δq and δp are small an attempt is made to develop an approximate method of solutions (quasi-orthodox approximation). Energy spectrum of an electron in a hydrogen atom is found in the second order of this approximation.

1. *Introduction*

Among the discussed aspects of the modern quantum theory the so-called 'incompleteness of the probability interpretation' occupies one of the most important places. One can explain this notion in an elementary fashion as follows.

It is well known that quantum mechanics is a statistical theory which deals with probabilities. Thus for any instant t quantum mechanics gives a probability coordinate density $\alpha(q,t) \geqslant 0$ and a probability momentum density $\beta(p,t) \geqslant 0$ correlated by the wave function ψ (or by the density matrix ρ). Let us suppose now that one of these physical variables, for example q, will be measured at an instant t'. The value $q(t')$ will naturally appear to be equal to q', one of the coordinate values allowed by the distribution $\alpha(q,t')$. Then any consistent probabilistic theory must give a

† On leave of absence from Peoples' Friendship University, Chair of Theoretical Physics, 3, Ordjonikidze Street, B-302, Moscow, U.S.S.R.

conditional probability of momentum, i.e. a probability momentum density $\beta'(p/q(t')=q')$ under the condition that the value q' of the coordinate is realised at the instant t'. However conditional probabilities do not enter into modern quantum mechanics, furthermore one is forbidden to think about such probabilities.

This inconsistency of quantum mechanics (from the point of view of probability theory) leads to a number of important consequences. For example, (1) it is not permitted to suppose that at a certain instant a quantum system can be found at a point $\{q,p\}$ in phase-space, (2) any attempt of model ideas about quantum systems is rejected, (3) the possibility of creating a theory giving a more complete description of the evolution of quantum mechanical systems seems to be doubtful, etc. These consequences have been the subject of continuous discussions during more than 40 years.

The interdictions mentioned above would apparently have been removed and many of the discussed problems would have been automatically resolved if quantum mechanics gave a phase-space probability density $F(q,p,t) \geqslant 0$ satisfying the usual conditions:

$$\int F(q,p,t)\,dp = \alpha(q,t), \qquad \int F(q,p,t)\,dq = \beta(p,t) \tag{1.1}$$

However the repeated attempts to introduce such a function in quantum mechanics turned out to be unsuccessful (Wigner, 1932; Terletsky, 1937; Blokhintsev, 1940; Margenau & Hill, 1961; Mehta, 1964; Cohen, 1966a; Shankara, 1967; Kuryshkin, 1968, 1969a). The quantum distribution functions $F(q,p,t)$ constructed by different authors appeared to be either real with a variable sign, or complex. The only non-negative F (Bopp, 1956) leads to a divergence with experimental data.

Finally it was proved that it was impossible to introduce a non-negative phase-space distribution function in modern quantum mechanics (Cohen, 1966b; Kuryshkin, 1969b). But it follows from this proof that it is possible to create a new quantum mechanics possessing the mentioned function. Such a theory was proposed by the author of this paper (Kuryshkin, 1969b, 1969c, 1971).

The preliminary investigations (Kuryshkin, 1969b, 1972a) show that the obtained theory is closed and self-consistent. Besides it contains the classical statistics and the essential part of the generally accepted quantum mechanics as particular limiting cases.

However the mathematical formalism of such a quantum mechanics includes a set of auxiliary functions of coordinate and time of which the meaning is not yet clear. These circumstances deprive us of the possibility of verifying the theory by the precise solution of concrete problems and the performance of the corresponding experiments.

That is why in the present paper we intend first to investigate the physical consequences due to the presence of auxiliary functions and second, to formulate more or less consistent approximate methods for the solution of concrete problems.

2. *Formulation of a Problem Within the Quantum Mechanics Possessing a Non-Negative Phase-Space Distribution Function*

Let a mechanical system be characterised (from the point of view of the classical theory) by a number of physical variables $A(q,p,t)$ given as functions of phase-space $\{q(q_1,q_2,\ldots,q_N),\ p(p_1,p_2,\ldots,p_N)\}$ and time t.

To formulate the corresponding quantum problem we shall choose some set of square integrable functions of the coordinates and time $\phi_k(q,t)$, $k = 1, 2, 3, \ldots$, normalised as follows:

$$\int \sum_k |\phi_k(q,t)|^2 \, dq = 1 \tag{2.1}$$

Constructing the auxiliary function of phase-space and time

$$\phi(q,p,t) = (2\pi\hbar)^{-N/2} \exp\left(-\frac{i}{\hbar}(qp)\right) \sum_k \phi_k(q,t)\, \tilde{\phi}_k^*(p,t), \tag{2.2}$$

where (qp) is the scalar product of the vectors q and p,

$$\phi_k(p,t) = (2\pi\hbar)^{-N/2} \int \phi_k(q,t) \exp\left(-\frac{i}{\hbar}(qp)\right) dq, \tag{2.3}$$

we shall define the action of the operator $O(A)$ corresponding to the function $A(q,p,t)$ on an arbitrary function $U(q,t)$ of coordinate and time by the equality

$$O(A)\, U(q,t) =$$
$$(2\pi\hbar)^{-N} \int \phi(\xi,\eta,t)\, A(q+\xi, p+\eta, t) \exp\left(\frac{i}{\hbar}((q-q')p)\right) U(q',t)\, d\xi\, d\eta\, dp\, dq' \tag{2.4}$$

The definition (2.4) represents a correspondence rule between the classical functions $A(q,p,t)$ and their quantum operators $O(A)$. The properties of the operators thus obtained have been studied in the papers (Kuryshkin, 1971, 1972a).

Having determined all the operators $O(A)$ we may use the conventional mathematical formalism of quantum mechanics: (1) the state of the system can be given by any normalised wave function $\psi(q,t)$ satisfying the equation

$$i\hbar \frac{\partial \psi(q,t)}{\partial t} = O(H)\, \psi(q,t) \tag{2.5}$$

(2) the expectation value $\langle A \rangle$ of a physical variable A in the state ψ is determined by the formula

$$\langle A \rangle = \int \psi^*(q,t)\, O(A)\, \psi(q,t)\, dq \tag{2.6}$$

Here $O(H)$ and $O(A)$ are the operators corresponding to the Hamilton function $H(q,p,t)$ of the system and the physical variable $A(q,p,t)$ due to the correspondence rule (2.4).

With the help of the relations (2.2–2.4) we easily obtain from (2.6)

$$\langle A \rangle = \int A(q,p,t)\, F(q,p,t)\, dq\, dp, \tag{2.7}$$

where the phase-space distribution function F is related to the wave function ψ by the formula

$$F(q,p,t) = (2\pi\hbar)^{-N} \sum_k \left| \int \phi_k(q-\xi,t)\, \psi^*(\xi,t) \exp\left(\frac{i}{\hbar}(\xi p)\right) d\xi \right|^2 \tag{2.8a}$$

The properties of the distribution function (2.8a) and its equation (Kuryshkin, 1969b, 1972a) permit us to treat F as the phase-space probability density, i.e. to give the probabilistic interpretation to the proposed theory (Kuryshkin, 1972a).

Integrating (2.8a) we obtain the relations analogous to (1.1)

$$\alpha(q,t) = \int |\psi(\xi,t)|^2 \sum_k |\phi_k(q-\xi,t)|^2\, d\xi \tag{2.8b}$$

$$\beta(p,t) = \int \left| (2\pi\hbar)^{-N/2} \int \psi(q,t) \exp\left(-\frac{i}{\hbar}(q\eta)\right) dq \right|^2 \sum_k |\tilde{\phi}_k(p-\eta,t)|^2\, d\eta \tag{2.8c}$$

which give us the probability densities of the coordinates and momenta (Kuryshkin, 1972a).

If $\alpha(q',t') \neq 0$ then the value q' of the coordinate can be realized at the instant t'. In this case we have

$$\beta'(p/q(t') = q') = \frac{1}{\alpha(q',t')} F(q',p,t) \tag{2.9}$$

i.e. the proposed quantum mechanics is free from the 'incompleteness of the probability interpretation'.

The formulas (2.5, 2.6, 2.8, 2.9) can be easily generalised on mixture of states given by a density matrix ρ (Kuryshkin, 1971, 1972a).

3. *The Simplest Quantum-Mechanical Problems*

In the preceding paragraph we have stated the mathematical formalism of quantum mechanics with a phase-space probability density. Nevertheless we are unable to apply this formalism to concrete quantum problems since we do not know the explicit form of the auxiliary functions. This is natural since the appearance of $\phi_k(q,t)$ in the theory is not connected with any physical reason but with certain mathematical demands (Kuryshkin, 1971).

To clarify the physical consequences due to the auxiliary functions we shall consider some problems permitting the exact solution with an arbitrary set of $\phi_k(q,t)$.

First of all we shall notice that our operators $O(A)$ corresponding to the

functions $A(q,p,t)$ which are integral and rational with respect to p, take the differential form:

$$O(A) = \int \phi(\xi,\eta,t)\, A(q+\xi, \hat{p}+\eta, t)\, d\xi\, d\eta \tag{3.1}$$

where

$$\hat{p}_j = -i\hbar \frac{\partial}{\partial q_j}, \qquad j = 1,2,\ldots,N. \tag{3.2}$$

The function A in (3.1) must be written down so that $\hat{p}_j$ do not operate on A. Introducing the notation

$$\overline{A(q,p,t)}^{\phi} = \int A(q,p,t)\, \phi(q,p,t)\, dq\, dp \tag{3.3}$$

which will be useful in the future if we obtain from (3.1) the coordinate and momentum operators:

$$O(q) = q + \bar{q}^{\phi}, \qquad O(p) = \hat{p} + \bar{p}^{\phi} \tag{3.4}$$

Let us start now with the treatment of the simplest quantum-mechanical problems.

1. *A free system* with Hamiltonian

$$H(q,p,t) = \sum_{j=1}^{N} \frac{1}{2m_j} p_j^2 \tag{3.5}$$

Taking into account the relations (2.1), (3.3) and (3.4) we have from (3.1) the total energy operator

$$O(H) = \sum_{j=1}^{N} \frac{1}{2m_j} \{O^2(p_j) + \bar{p}_j^{2\phi} - (\bar{p}_j^{\phi})^2\} \tag{3.6}$$

The eigenfunctions of this operator are

$$\psi_P = L_q^{-N/2} \exp\left(\frac{i}{\hbar}(qP)\right), \tag{3.7}$$

with the eigenvalues

$$E_P = \sum_{j=1}^{N} \frac{1}{2m_j} \{P_j^2 + 2\bar{p}_j^{\phi} P_j + \bar{p}_j^{2\phi}\} \geqslant 0 \tag{3.8}$$

where $P = P(P_1, P_2, \ldots, P_N)$ and the coefficient

$$L_q^N = \int dq = (2\pi\hbar)^N \delta_p(0)$$

is introduced to have the formal normalisation of $|\psi|^2$ to unity.

Having determined the operators corresponding to the integral and rational functions of momentum we obtain according to (2.6)

$$\left\langle \prod_{j=1}^{N} p_j^{n_j} \right\rangle_P = \sum_{l_1=0}^{n_1} \cdots \sum_{l_N=0}^{n_N} c_{n_j}^{l_j} P_j^{n_j - l_j} \left(\overline{\prod_{j=1}^{N} p_j^{l_j}} \right)^{\phi} \tag{3.9}$$

c_n^l being the binomial coefficients. Hence it follows in particular

$$\langle p_j\rangle_P = P_j + \bar{p}_j^{\phi}, \qquad \langle p_j{}^2\rangle_P = P_j{}^2 + 2\bar{p}_j^{\phi} P_j + \bar{p}_j^{2\phi} \tag{3.10}$$

which jointly with (3.8) gives the usual relation between $\langle E\rangle$ and $\langle p_j{}^2\rangle$.

Finally combining (2.8) and (2.6) we arrive at the normalised probability density

$$F_P(q,p,t) = L_q^{-N} \sum_k |\tilde{\phi}_k(p-P,t)|^2 \tag{3.11a}$$

$$\alpha_P(q,t) = L_q^{-N} \tag{3.11b}$$

$$\beta_P(p,t) = \sum_k |\tilde{\phi}_k(p-P,t)|^2 \tag{3.11c}$$

Hence when the state of a free system is an eigenfunction of the energy operator, the momentum not precisely determined but distributed with the probability density (3.11c). One can easily verify that the statistical average of momentum functions over the distribution (3.11c) coincides with the results (3.9).

By constructing the operators $O(H^n)$ and calculating $\langle E^n\rangle_P$ one can find the energy distribution and show that the energy is not precisely determined.

2. *The minimum uncertainty of momentum.* Having constructed the operator $O((\Delta p_j)^2)$ where $\Delta p_j = p_j - \langle p_j\rangle$ and calculated its expectation value in an arbitrary state ψ after a simple transformation we shall have

$$\langle(\Delta p_j)^2\rangle = \langle(O(p_j) - \langle p_j\rangle)^2\rangle + [\bar{p}_j^{2\phi} - (\bar{p}_j^{\phi})^2] \geqslant 0$$

Both terms on the right-hand side of this expression are non-negative. Hence with any set of the auxiliary functions $\phi_k(q,t)$ the minimum uncertainty of momentum

$$\min_{\Psi}\{\langle(\Delta p_j)^2\rangle\} = \bar{p}_j^{2\phi} - (\bar{p}_j^{\phi})^2 \stackrel{\text{def}}{=} (\delta p_j)^2 \geqslant 0 \tag{3.12}$$

is attained when the wave function satisfies the equation

$$O(p)\psi = \langle p\rangle\psi \tag{3.13}$$

Thus independently of the number and the explicit form of ϕ_k the momentum is mostly determined in the p-states (there is some difference here between the p- and the $\hat{p}$-states since generally $O(p) \neq \hat{p}$).

3. *The p-states.* Since the functions (3.7) are eigenfunctions of the operator $O(p)$ with the eigenvalues $\langle p\rangle = P + \bar{p}^{\phi}$ the equation (3.13) leads to results completely coinciding with the formulas (3.9–3.11). Thus (3.11c) is the momentum distribution in the states where the momentum is mostly determined.

4. *The q-states.* The equation

$$O(q)\psi = \langle q\rangle\psi \tag{3.14}$$

gives us the normalised eigenfunctions

$$\psi_Q(q) = L_p^{-N/2}(2\pi\hbar)^{-N/2}\,\delta(q-Q) \tag{3.15}$$

with the eigenvalues $\langle q\rangle_Q = Q + \bar{q}^{\phi}$, $Q = Q(Q_1, Q_2, \ldots, Q_N)$.

In this case the expectation values of coordinate functions are given by formulas analogous to (3.9) and (3.10) (one has to substitute $p \to q, P \to Q$) and the normalised probability densities have the forms

$$F_Q(q,p,t) = L_p^{-N} \sum_k |\phi_k(q-Q,t)|^2 \tag{3.16a}$$

$$\alpha_Q(q,t) = \sum_k |\phi_k(q-Q,t)|^2 \tag{3.16b}$$

$$\beta_Q(p,t) = L_p^{-N} \tag{3.16c}$$

5. *The minimum uncertainty of the coordinate.* Having constructed the operator $O((\Delta q_j)^2)$, $\Delta q_j = q_j - \langle q_j \rangle$, we find that the minimum coordinate uncertainty

$$\min_{\Psi} \{\langle (\Delta q_j)^2 \rangle\} = \bar{q}_j^{2\phi} - (\bar{q}_j^{\phi})^2 \stackrel{\text{def}}{=} (\delta q_j)^2 \geqslant 0 \tag{3.17}$$

is attained in the states ψ satisfying the equation (3.14). Hence with any set of $\phi_k(q,t)$ the coordinate is mostly determined in the q-states. The expression (3.16b) gives us the probability density of coordinate in such states.

Thus the quantum mechanics with a non-negative phase-space distribution function generally leads to 'fluctuations' of the energy, momentum and coordinate in the E-, p- and q-states while the expectation values, dispersions and the corresponding distributions are functionals of the auxiliary functions.

4. *The Notion of a 'Subquantum Situation'*

The results of the preceding paragraph might be easily comprehended if one supposes that the auxiliary functions $\phi_k(q,t)$ reflect some unknown (or perhaps already partially known) physical reality.

Later on following the accepted terminology (Kuryshkin, 1972b) we shall say that the functions $\phi_k(q,t)$ represent some 'subquantum situation' in which the considered system is placed. By introducing this special term we separate the physical reality reflected in our theory by the auxiliary functions from the mechanical system placed in this reality (and interacting with it). In other words the 'subquantum situation' can exist apart from any mechanical system. On the other hand different systems can be placed in the same 'subquantum situation'.

We do not principally use the presently well-known term 'hidden parameters'. In fact from the point of view of our theory the considered mechanical systems possess classical (and only classical) parameters. But its behaviour is non-classical (quantum). This might be explained by some hidden properties of the system itself as well as by its interaction with the 'subquantum situation'. Naturally this interaction might be carried out by means of some 'hidden parameters' inherent in the system as well as in the 'subquantum situation'.

The classical characteristics of the mechanical system (i.e. the totality of the functions $A(q,p,t)$) and the 'subquantum situation' (the set of $\phi_k(q,t)$) completely determine the operators and consequently the solutions of the considered quantum-mechanical problem. The treatment of the same system in another 'subquantum situation' will lead to other solutions. This is natural since the 'subquantum situation' interacts with the system which is placed in it. In accordance with the results obtained above the 'subquantum situation' determines the momentum distribution (3.11c) of a 'free' system, the coordinate distribution (3.16b) of a 'fixed' system as well as the minimum coordinate (3.17) and momentum (3.12) dispersions. It is quite clear that the 'subquantum situation' will also give a contribution when the states of a system differ from those examined above (see for example the harmonical occilator (Kuryshkin, 1972)).

The coordinate and time dependence of ϕ_k allows to make a number of suppositions about the physical nature of the 'subquantum situation' such as: the physical properties of space-time, physical 'vacuum', physical environment (the apparatus for example) and so on. These suppositions are not new. Some of them are already worked out rather well and apparently are similar in their physical meaning to our 'subquantum situation', for example the 'subquantum medium' (De Broglie, 1964, 1968; Bohm & Vigier, 1954; Terletsky, 1960).

In this paper we are not going to discuss the possible physical interpretations of the 'subquantum situations' since first of all we have to clarify the following:

1. Whether results of the proposed theory correspond to the known experimental data, and
2. If they do, is it possible to verify the existence of the 'subquantum situation' experimentally?

These questions are not easy to answer. As a rule the problems resolved with an arbitrary 'subquantum situation' do not permit to perform an experimental test while we are unable to resolve a problem tested by experiment in the most general case. Thus there appears the necessity to restrict the totality of the 'subquantum situations'. In other words one has to formulate some other (in addition to (2)) mathematical requirements on ϕ_k (some physical proposals about the 'subquantum situation') in order to simplify the solutions of concrete problems without loss of generality of their formulations.

5. *Stationary Homogeneous Isotropic 'Subquantum Situation'*

To simplify the concrete problem solutions we shall require the 'subquantum situation' to be stationary in time, homogeneous and isotropic in the 3-dimensional space.

It is necessary to give a detailed explanation of this requirement. We suppose that the interaction of the 'subquantum situation' with a system

placed in it does not depend on the instant, orientation and region of location of the system in 3-dimensional space.

At first sight it seems that we have identified the 'subquantum situation' with something like an ether uniform over the universe. But this is not true. Since our theory is probabilistic there always exists a region in which the probability of the system stay is close to unity. We suppose that the 'subquantum situation' is the same at any point of this region. It is quite possible that the 'subquantum situation' in other regions is essentially different. However it is unimportant since the probability that the system hits such a region is nearly zero. This fact allows us to spread the 'subquantum situation' from the region of the system stay to the whole universe. As far as the stationarity is concerned our supposition means that in the region of the system stay the 'subquantum situation' does not change during the period of observation. Its change in this region during another period of time and in other regions of the space in this period is not essential for the investigated problem.

From a mathematical point of view one can write down the requirement of the stationarity, homogeneity and isotropy of the 'subquantum situation' as follows:

$$\phi_k(\mathbf{r}, t) = \phi_k(r), \qquad k = 1, 2, \ldots \tag{5.1}$$

where $\mathbf{r} = \mathbf{r}(x, y, z)$ is a point of the three-dimensional space, $r = |\mathbf{r}|$.

The stationarity of the 'subquantum situation' (5.1) is obvious. We shall show now that it is also homogeneous and isotropic.

Let the coordinate q of a system be mostly determined. Then (see the results of Section 3) the expectation value $\langle q \rangle$ can be arbitrary and the coordinate probability density in accordance with (3.16b) is:

$$\alpha(q, t) = \sum_k |\phi_k(q - \langle q \rangle + \bar{q}^{\phi}, t)|^2 \tag{5.2}$$

Let the system consist of a single particle. Then in the 'subquantum situation' (5.1) $\bar{\mathbf{r}}^{\phi} = 0$ and it follows from (5.2)

$$\alpha(\langle \mathbf{r} \rangle + \boldsymbol{\xi}, t) = \sum_k |\phi_k(|\boldsymbol{\xi}|)|^2 \tag{5.3}$$

where $\boldsymbol{\xi} = \mathbf{r} - \langle \mathbf{r} \rangle$. Thus in the 'subquantum situation' (5.1) the probability of finding the particle at a distance ξ from its expectation coordinate $\langle \mathbf{r} \rangle$ depends neither on time (stationary), nor on the coordinate $\langle \mathbf{r} \rangle$ of the particle (homogeneity) but on the modulus of the deviation ξ (isotropy).

The treatment of a free particle in the 'subquantum situation' (5.1) leads to analogous results. In fact combining (2.3) and (5.1) we have

$$\phi(\mathbf{p}, t) = -\frac{2}{p\sqrt{2\pi\hbar}} \int_0^{\infty} r\phi_k(r) \sin\frac{rp}{\hbar} dr = \tilde{\phi}_k(p) \tag{5.4}$$

where $\mathbf{p} = \mathbf{p}(p_x, p_y, p_z)$ is a 3-dimensional momentum, $p = |\mathbf{p}|$. Together with (3.10) and (3.11b) this gives us

$$\beta(\langle \mathbf{p} \rangle + \boldsymbol{\eta}, t) = \sum_k |\tilde{\phi}_k(|\boldsymbol{\eta}|)|^2 \tag{5.5}$$

i.e. the probability of a 'subquantum' addition η to the momentum $\mathbf{p}$ depends neither on time nor on the momentum of the particle. These results indicate once more the space homogeneity and isotropy of the 'subquantum situation' (5.1).

Now we have to decide what is the 'subquantum situation' $\phi_k(q,t)$ in a configurational space $q=q(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_n)$, which arrives when a system consists of n objects. Since all of these objects are in the same three-dimensional space, i.e. each object is placed into the same 'subquantum situation' $\phi_k(r)$, $\phi_k(q,t)$ consists of $\phi_k(r_i)$; $i=1, 2,\ldots, n$. The functions $\phi_k(q,t)$ must be symmetric with respect to all permutations of r_i, since the interaction between the isotropic 'subquantum situation' and a system does not depend on the orientation of the latter. Besides one has to keep the formalization (2.1). It is therefore natural to write down

$$\phi_k(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_n)=\left\{\int|\phi_k(r)|^2\,d\mathbf{r}\right\}^{(n-1/2)}\phi_k(r_i)\ldots\phi_k(r_n) \tag{5.6}$$

It follows from (5.6) that the 'subquantum situation' leads to some dependence (correlation) of classically independent systems which is natural. We shall not prove or discuss this important consequence in the present paper. We shall only mention that in our theory one must be very careful in dividing a system into several classically independent parts.

It is interesting to notice that correlations of the classically independent systems sometimes take place also in the generally accepted quantum mechanics (Einstein *et al.*, 1933).

6. *Approximate Methods*

The quantum mechanics with a non-negative phase-space distribution function has two types of mathematical formalism (Kuryshkin, 1972a). But at the present time the operational formalism seems to be preferable. Firstly, the equation for ψ is more simple then that for F, in spite of the fact that both of them are integro-differential. Secondly, the solution of problems in the operational formalism provides the opportunity to use the rich mathematical apparatus of the generally accepted quantum mechanics. Thirdly, the operator $O(A)$ of an integral and rational (with respect to momentum) function $A(q,p,t)$ can be written down in the differential form.

We shall rewrite here the definition (3.1) in the equivalent form:

$$O(A(q,p,t))=\left\{\int\phi(\xi,\eta,t)\,A(q+\xi,p+\eta,t)\,d\xi\,d\eta\right\}_{p=\hat{p}} \tag{6.1}$$

where the curly brackets mean the regulation of the function with respect to momentum, i.e. the placement of p_j after the coordinate functions before the substituting p_j by $\hat{p}_j$.

For the following calculations one has to choose the explicit form of $\phi_k(q,t)$. However it is possible to consider some successive approximations based on the following reasonings.

In our theory the momentum and coordinate dispersions have inferior limits in any state:

$$\langle(\Delta q_j)^2\rangle \geqslant (\delta q_j)^2, \qquad \langle(\Delta p_j)^2\rangle \geqslant (\delta p_j)^2 \tag{6.2}$$

where $(\delta q_j)^2$ and $(\delta p_j)^2$ are the 'subquantum' uncertainties. In the case of stationary homogeneous isotropic 'subquantum situation' (5.6) they do not depend on the component number:

$$\delta q_j = \sqrt{\left(\int q_j^2 \sum_k |\phi_k(q)|^2 \, dq\right)} = \delta q \tag{6.3}$$

$$\delta p_j = \sqrt{\left(\int p_j^2 \sum_k |\tilde{\phi}_k(p)|^2 \, dp\right)} = \delta p \tag{6.4}$$

which follows from the relations (2.2), (3.12), (3.17), (5.4) and (5.6).

The limitations (6.2) permit us to make the three following suppositions about the 'subquantum situation'.

1. *δq is small.* Then it follows from (6.3) that all $\phi_k(q)$ (and also $\phi(q,q)$) differ essentially from zero only for $q \sim 0$. Hence we can use in (6.1) the Taylor series development of A and write down the operator of accordinate function in the form of successive approximations

$$\begin{aligned} O(V(q,t)) &= \int \mathrm{e}^{(\xi\Delta_q)} \phi(\xi,\eta)\, d\xi\, d\eta\, V(q,t) \\ &= \int \{1 + (\xi\nabla_q) + \tfrac{1}{2}(\xi\nabla_q)^2 + \ldots\}\, \phi(\xi,\eta)\, d\xi\, d\eta\, V(q,t) \end{aligned} \tag{6.5}$$

where

$$\nabla_q = \left\{\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots, \frac{\partial}{\partial q_N}\right\}$$

2. *δp is small.* An analogous reasoning with the help of (3), (6.1) and (6.4) give the successive operators corresponding to a momentum functions

$$O(g(p,t)) = \left\{\int \mathrm{e}^{(\eta\nabla_p)}\, \phi(\xi,\eta)\, d\xi\, d\eta\, g(p,t)\right\}_{p=\hat{p}} \tag{6.6}$$

3. *δq and δp are small.* Then both of the series (6.5) and (6.6) are correct and furthermore

$$O(A(q,p,t)) = \left\{\int \exp(\xi\nabla_q) + (\eta\nabla_p)\, \phi(\xi,\eta)\, d\xi\, d\eta\, A(q,p,t)\right\}_{p=\hat{p}} \tag{6.7}$$

In the case for functions of the form $f(q) + g(p)$ in the zero-order approximation we have the orthodox operators

$$O(g(p,t) + f(q,t)) = g(\hat{p},t) + f(q,t)$$

Therefore later on we shall use the term 'quasi-orthodox approximation' for the approximate method based on the third supposition about the 'subquantum situation'.

We shall notice one difficulty in the series (6.7). To determine the order

of the smallness of a term in these series is not easy. First, the orders of smallness of δq and δp can be different. Second, the correlative terms of the type $(\xi\eta)^n$ can be rather large. However this situation will be simplified by the fact that for the concrete operators the series (40) will be finite because the orders of p in the real physical variables are not large.

At last we shall explain how the expressions 'δq is small' and 'δp is small' are understood. The quantities δq and δp are denominate and their absolute values depend on the choice of the system of units. Therefore by saying for example 'δq is small' we mean that

$$\delta q \ll \Delta q = \min_j \{\Delta q_j\} \tag{6.8}$$

where

$$\Delta q_j = \sqrt{(\langle(\Delta q_j)^2\rangle)}, \qquad j = 1, 2, \ldots, N$$

In other words the 'subquantum' coordinate uncertainty δq is much less than the coordinate uncertainty Δq of the system placed in this 'subquantum situation'. Hence it is possible to justify the correctness of a chosen approximate method only after having resolved the problem (for the calculation of Δq_j one must already know the state ψ). If for example $\delta q \simeq 10^{-11}$ cm and as a result one has $\Delta q \simeq 10^{-8}$ cm then the supposition about the smallness of δq might be considered justified. But if Δq turns out to be less than 10^{-11} cm the quasi-orthodox approximation is doubtful and one has to resolve the problem beginning with the redetermination of operators. In this case one must naturally know the character of the 'subquantum situation', i.e. the explicit from of all $\phi_k(r)$.

7. *Energy Spectrum of an Electron in the Coulomb Potential. The Second Order of the Quasi-Orthodox Approximation*

Let a particle of a mass μ be in the potential field $V(\mathbf{r})$. Then the main variables characterising this particle are the following: coordinate $\mathbf{r}$, momentum $\mathbf{p}$, moment $\mathbf{L} = [\mathbf{r} \times \mathbf{p}]$, kinetic energy $T = p^2/2\mu$, potential energy $V = V(\mathbf{r})$, total energy $E = T + V$.

In a stationary homogeneous isotropic 'subquantum situation' (5.1) we have from (6.5–6.7) the following operators:

$$\left.\begin{aligned} O(\mathbf{r}) &= \mathbf{r}, \qquad O(\mathbf{p}) = -i\hbar\nabla, \qquad O(\mathbf{L}) = -i\hbar[\mathbf{r} \times \nabla] \\ O(r^2) &= r^2 + 3(\delta r)^2, \qquad O(p^2) = -\hbar^2\nabla^2 + 3(\delta p)^2 \end{aligned}\right\} \tag{7.1}$$

$$O(T) = -\frac{\hbar^2}{2\mu}\nabla^2 + \frac{3}{2\mu}(\delta p)^2 \tag{7.2}$$

$$O(V) = V(\mathbf{r}) + \tfrac{1}{2}(\delta r)^2\nabla^2 V(\mathbf{r}) + \cdots \tag{7.3}$$

where δr and δp are the 'subquantum' uncertainties of the coordinate and momentum. The operators (7.1–7.2) are exact, the operator (7.3) is given in the second order with respect to δr. The operator $O(E)$ is the sum of (7.2) and (7.3).

Let the considered particle be an electron (charge $-e$) in the Coulomb field (a charge $+Ze$ placed in the point $\mathbf{r} = 0$). Then

$$V(\mathbf{r}) = -\frac{Ze^2}{r}, \qquad \nabla^2 V(\mathbf{r}) = 4\pi Ze^2 \delta(\mathbf{r}). \tag{7.4}$$

Combining (7.2), (7.3) and (7.4) we have the total energy operator in the second order of the quasi-orthodox approximation:

$$O(E) = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{Ze^2}{r} + \frac{3(\delta p)^2}{2\mu} + 2\pi Ze^2(\delta r)^2 \delta(\mathbf{r}) \tag{7.5}$$

It is now natural to treat the problem in accordance with perturbation theory by considering the last term in (7.5) as the perturbation operator. For the non-perturbed equation

$$\left\{-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{Ze^2}{r} + \frac{3}{2\mu}(\delta p)^2\right\}\psi(\mathbf{r}) = E^{(0)}\psi(\mathbf{r})$$

we obtain the eigenfunctions $\psi_{nlm}(\mathbf{r})$ coinciding with those of the corresponding problem in the generally accepted quantum mechanics (Fermi, 1961) and the spectrum of eigenvalues

$$E_n^{(0)} = -\frac{Z^2 e^2}{2an^2} + \frac{3}{2\mu}(\delta p)^2$$

where

$$n = 1, 2, \ldots; \qquad l = 0, 1, \ldots, n-1; \qquad m = -l, \ldots, 0, \ldots, l,$$

$$a = \hbar^2/\mu e^2$$

The eigenvalue $E_n^{(0)}$ is n^2-fold degenerate but the matrix of the perturbation operator is diagonal since

$$\langle nlm|\delta(\mathbf{r})|nlm\rangle = \frac{Z^3}{\pi a^3 n^3}\delta_{l0}$$

Therefore the secular equation for the nth eigenvalue gives only one (the degeneracy is not completely removed) but totally determined correction, i.e. the only eigenvalue which increases corresponds to the eigenfunction ψ_{n00}. Finally we have:

$$E_{nl} = -\frac{Z^2 e^2}{2an^2} + \frac{3}{2\mu}(\delta p)^2 + \frac{2Z^4 e^2(\delta r)^2}{a^3 n^3}\delta_{l0} \tag{7.6}$$

The first term in (7.6) represents the energy levels coinciding with those obtained in the generally accepted quantum mechanics. The second term giving a positive addition to the energy, does not influence the difference between the levels. Hence it cannot be experimentally verified. The third term indicates a shift of the S-levels which must lead to a splitting of the spectral lines.

Such a shift is observed experimentally (Lamb & Retherford, 1947) and has been qualitatively explained in the quantum electrodynamics (Bethe, 1947). In accordance with experiment the relative shift of $2S_{1/2}$ and $2P_{1/2}$ energy levels of electron in a hydrogen atom is approximately 9% (0·033 cm^{-1}) of the fine structure, i.e.:

$$\Delta E \simeq 0{\cdot}09 \frac{\alpha^2 c^2}{32a} \tag{7.7}$$

where $\alpha = e^2/\hbar c$ is the fine structure constant.

Comparing (7.7) with our result when $z = 1$, $n = 2$ we get an estimation of the 'subquantum' coordinate uncertainty

$$\delta r \simeq 0{\cdot}11 \alpha a \simeq 4{\cdot}1 \,.\, 10^{-12} \text{ cm} \tag{7.8}$$

In order to evaluate the 'subquantum' momentum uncertainty we shall suppose that $E_n < 0$ for the experimentally observed states. Then (7.6) gives us the relation between δp and the maximum level number $n_{\max}$:

$$\sqrt{3}\, n_{\max} \delta p < \alpha Z \mu c \tag{7.9}$$

Since for the hydrogen atom ($Z = 1$) $n_{\max}$ is not less than 30 we obtain from (7.9)

$$\delta p < 0{\cdot}02 \alpha \mu c \simeq 3{\cdot}9 \,.\, 10^{-21} \text{ g cm/sec} \tag{7.10}$$

To justify the quasi-orthodox approximation we shall calculate the coordinate and momentum dispersion in the ψ_{nlm}-state:

$$\langle (\Delta \mathbf{r})^2 \rangle = \frac{n^2 a^2}{2Z^4} [5n^2 + 1 - 3l(l+1)]$$

$$\langle (\Delta \mathbf{p})^2 \rangle = \frac{\mu Z^2 e^2}{an^2} + 3(\delta p)^2$$

Considering $\langle (\Delta \mathbf{r})^2 \rangle = 3(\Delta r)^2$, $\langle (\Delta \mathbf{p})^2 \rangle = 3(\Delta p)^2$ and using (7.8–7.9) we obtain

$$\frac{\delta r}{\Delta r} \leqslant 0{\cdot}0014 \frac{Z^2}{n^2}, \qquad \frac{\delta p}{\Delta p} = \left(1 + \frac{n_{\max}^2}{n^2}\right)^{-1/2}$$

Hence one can consider the δq-approximation to be justified. The δp-approximation is good only for small n. We must point out that the δp-approximation was not explicitly used in the above calculations.

8. *Conclusions*

In spite of the rather hopeful results of the preceding paragraph we are still far from the statement that quantum mechanics with a non-negative phase-space distribution function is adequate. In fact we have not yet answered the main questions posed in the fourth paragraph.

The only new result tested by experiment (the energy shift of the S-states) cannot be treated as a proof of the correctness of the proposed theory, since the same result can be obtained in different ways (Bethe, 1947; Welton, 1948).

On the other hand the proposed quantum mechanics gives rise to a number of interpretational and mathematical difficulties. The first consists of the fact that such a quantum mechanics demands a new interpretation and consequently a reconsideration of modern views on the quantum theory. But for the present we do not consider this problem to be the most important. First of all we must find out whether this theory leads to correct results. Here appears the main difficulty, which is of mathematical nature.

The solution of the first problem verified by experiment shows that the variable $\delta q \delta p$ which one can consider as a sort of action of the 'subquantum situation' is very small. That follows from the relations (7.8) and (7.10):

$$\delta r . \delta p < 0{\cdot}0022\alpha\hbar \simeq 1{\cdot}6 . 10^{-5}\hbar \tag{8.1}$$

(remember the three basic suppositions made in the preceding paragraph: the third term in (7.6) represents the Lamb shift, only 30 energy levels of the electron exist in the hydrogen atom, the energy of this level is negative).

It is possible to show that the result (8.1) is not accessible while the functions $\phi_k(q)$ are analytic. But the inequality (8.1) can be easily satisfied choosing a set of non-analytic singular functions for which $|\phi_k(q)|^2$ is a generalised function (Kuryshkin, 1969). The mathematical theory of such functions is not yet sufficiently developed. Hence we are unable to perform the differentiation and the integration by parts which are necessary for more complicated calculations.

Thus at the present stage of development the quantum mechanics with a non-negative phase-space distribution function is not more than a formalism which needs a profound mathematical investigation.

Acknowledgements

The author expresses his sincerest appreciation for guidance and help from Professor Ya. P. Terletsky and gratefully acknowledges helpful discussions with Professors L. De Broglie, G. Lochak, V. D. Lyakhovsky, G. Petiau and J. P. Vigier.

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